Week 3

12 Explain why there does not exist a list of six polynomials that is linearly independent in $\mathcal{P}_4(\mathbf{F})$.

Solution: If the list does exist, say $f_1(x), \cdots, f_6(x) \in \mathcal{P}_4(\mathbb{F})$ linearly independent.

We assume $f_i(x) = a_{i0} + a_{i1}x + \cdots + a_{i4}x^4$, $a_{ij} \in \mathbb{F}$, $\forall\ 1 \le i \le 6,\ 0 \le j \le 4$.

If there exist $x_1, \cdots, x_6 \in \mathbb{F}$, such that $x_1 f_1(x) + \cdots + x_6 f_6(x) = 0$, then

$$\sum_{i=1}^{6} a_{i0}x_i + \left(\sum_{i=1}^{6} a_{i1}x_i\right)x + \cdots + \left(\sum_{i=1}^{6} a_{i4}x_i\right)x^4 = 0$$

$$\Rightarrow \sum_{i=1}^{6} a_{ij}x_i = 0, \quad \forall\ 0 \le j \le 4$$

$$\Rightarrow \underbrace{\begin{bmatrix} a_{10} & a_{20} & \cdots & a_{60} \\ a_{11} & a_{21} & \cdots & a_{61} \\ \vdots & \vdots & & \vdots \\ a_{14} & a_{24} & \cdots & a_{64} \end{bmatrix}_{5\times 6}} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_6 \end{bmatrix} = 0 \quad \cdots \quad (*)$$

This 5×6 matrix has rank at most 5, but the number of unknowns is 6, so $(*)$ has infinitely many solutions.

So $f_1(x), \cdots, f_6(x)$ are not linearly independent. #

7 Prove or give a counterexample: If v_1, v_2, v_3, v_4 is a basis of V and U is a subspace of V such that $v_1, v_2 \in U$ and $v_3 \notin U$ and $v_4 \notin U$, then v_1, v_2 is a basis of U.

Solution: Let $V = \mathbb{R}^4$, $v_1 = (1,0,0,0)$, $v_2 = (0,1,0,0)$, $v_3 = (0,0,1,0)$, $v_4 = (0,0,0,1)$

and let $U = \{(x,y,z,z) \in \mathbb{R}^4 : x,y,z \in \mathbb{R}\}$.

then $v_1, v_2 \in U$, $v_3 \notin U$, $v_4 \notin U$.

i.e. U can be spanned by v_1, v_2, v_3+v_4.

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6 (a) Let $U = \{p \in \mathcal{P}_4(\mathbf{F}) : p(2) = p(5)\}$. Find a basis of U.

(b) Extend the basis in part (a) to a basis of $\mathcal{P}_4(\mathbf{F})$.

(c) Find a subspace W of $\mathcal{P}_4(\mathbf{F})$ such that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.

Solution: (a). $\forall\ p(x) \in U$, Let $q(x) = p(x) - p(2) = p(x) - p(5)$,

then $q(2) = q(5) = 0 \Rightarrow \exists\ h(x) = ax^2+bx+c \in \mathcal{P}_2(\mathbb{F})$,

such that $q(x) = (x-2)(x-5)h(x)$

$$\Rightarrow p(x) = (x-2)(x-5)h(x) + p(2)$$
$$= a\underbrace{x^2(x-2)(x-5)} + b\underbrace{x(x-2)(x-5)} + c\underbrace{(x-2)(x-5)} + p(2)\cdot\underbrace{1}$$

thus we have $U = \text{span}\{x^2(x-2)(x-5),\ x(x-2)(x-5),\ (x-2)(x-5),\ 1\}$

(b). (c). It suffices to find a $p(x) \in \mathcal{P}_4(\mathbb{F})$, such that $p(x)$, $x^2(x-2)(x-5)$, $x(x-2)(x-5)$, $(x-2)(x-5)$, 1 are lin. indept.

Obviously, $p(x) = x \notin U$ and satisfies lin. indepce.

i.e. $\mathcal{P}_4(\mathbb{F}) = \text{span}\{x\} \oplus U$.

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Ex Show that

$\{e^{nx} : n \in \mathbb{Z}\} \subseteq \mathbb{R}^{\mathbb{R}}$ is lin indept

Hint: Differentiation and Vandermonde matrix

Solution: To prove $\{e^{nx} : n\in\mathbb{Z}\}$ is lin. indept. is to prove for any finite integer $N>0$, the finite subset $\{e^{nx} : n\in\mathbb{Z}, -N\le n\le N\}$ is lin. indept.

So if there exist $a_n \in \mathbb{R}$, $-N\le n\le N$, such that

$$f(x) \overset{\text{def.}}{=\!=} \sum_{n=-N}^{N} a_n e^{nx} = 0$$

$$\Rightarrow \quad f^{(k)}(x) = \sum_{n=-N}^{N} a_n \cdot n^k e^{nx} = 0, \quad \forall\ k\ge 0$$

$$\Rightarrow \quad f^{(k)}(0) = \sum_{n=-N}^{N} a_n \cdot n^k = 0, \quad \forall\ k\ge 0 \quad \cdots \quad (*_k)$$

[Or you can just use Taylor's expansion: $e^{nx} = \sum_{k=0}^{\infty} \frac{1}{k!}(nx)^k$. and the lin. independence of $1, x, x^2, \ldots, x^k, \ldots$ to achieve $(*_k)$.]

These infinitely many equations $(*_k)$ can be written in the form

$$\begin{array}{l} k=0 \longrightarrow \\ k=1 \longrightarrow \\ k=2 \longrightarrow \\ \vdots \\ \\ \\ \end{array} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ -N & (-N+1) & \cdots & N \\ (-N)^2 & (-N+1)^2 & \cdots & N^2 \\ \vdots & \vdots & & \vdots \\ (-N)^k & (-N+1)^k & \cdots & N^k \\ \vdots & \vdots & & \vdots \end{bmatrix} \begin{bmatrix} a_{-N} \\ a_{-N+1} \\ \vdots \\ a_{N-1} \\ a_N \end{bmatrix} = 0 \quad \cdots \quad (\star)$$

Theorem:

For the Vandermand matrix: $\begin{bmatrix} 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \alpha_m & \alpha_m^2 & \cdots & \alpha_m^{n-1} \end{bmatrix}_{m\times n}$

if $m \leq n$ and $\alpha_1, \cdots, \alpha_m$ are distinct, then it has full rank m.

So taking $k \geq 2N+1$, we have ~~that~~ the matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ -N & -N+1 & \cdots & N \\ (-N)^2 & (-N+1)^2 & \cdots & N^2 \\ \vdots & \vdots & & \vdots \\ (-N)^{k-1} & (-N+1)^{k-1} & \cdots & N^{k-1} \end{bmatrix}_{k\times(2N+1)}$$ has rank $2N+1$.

So $(\star)$ has the unique solution $a_n = 0$, $\forall -N \leq n \leq N$.

i.e. $\{ e^{nx} : n \in \mathbb{Z}, -N \leq n \leq N \}$

is lin. indept.

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